

ON RESTRICTED SUMSETS OVER A FIELD

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ABSTRACT. We consider restricted sumsets over field F . Let

$$C = \{a_1 + \cdots + a_n : a_1 \in A_1, \dots, a_n \in A_n, a_i - a_j \notin S_{ij} \text{ if } i \neq j\},$$

where S_{ij} ($1 \leq i \neq j \leq n$) are finite subsets of F with cardinality m , and A_1, \dots, A_n are finite nonempty subsets of F with $|A_1| = \cdots = |A_n| = k$. Let $p(F)$ be the additive order of the identity of F . It is proved that $|C| \geq \min\{p(F), n(k-1) - mn(n-1) + 1\}$ if $p(F) > mn$. This conclusion refines the result of Hou and Sun [11].

1. INTRODUCTION

Let F be a field. Denote by $p(F)$ the additive order of the identity of F . It is well-known that $p(F)$ is either infinite or a prime. For a finite set A , we use $|A|$ to denote the cardinality of A .

Suppose that A_1, \dots, A_n are finite nonempty subsets of F with $|A_j| = k_j$ for $1 \leq j \leq n$. The Cauchy-Davenport Theorem asserts that

$$|\{a_1 + \cdots + a_n : a_1 \in A_1, \dots, a_n \in A_n\}| \geq \min\{p(F), k_1 + \cdots + k_n - n + 1\}.$$

Let A be a finite subset of F . We define

$$n^{\wedge}A = \{a_1 + \cdots + a_n : a_1, \dots, a_n \in A, a_1, \dots, a_n \text{ are distinct}\}.$$

P. Erdős and H. Heilbronn [7] conjectured that

$$|2^{\wedge}A| \geq \min\{p(F), 2|A| - 3\}.$$

This conjecture was solved by Dias da Silva and Hamidoune [5], who established

$$|n^{\wedge}A| \geq \min\{p(F), n|A| - n^2 + 1\}.$$

In 1995-1996, Alon, Nathanson and Ruzsa [2, 3] developed the polynomial method to show if $0 < k_1 < k_2 < \cdots < k_n$, then

$$|\{a_1 + \cdots + a_n : a_i \in A_i, a_1, \dots, a_n \text{ are distinct}\}| \geq \min\{p(F), \sum_{j=1}^n (k_j - j) + 1\}.$$

Various restricted sumsets of A_1, \dots, A_n were investigated in [11], [12], [13], [14], [15] and [16]. In particular, Hou and Sun [11] considered the following sumset

$$C = \{a_1 + \cdots + a_n : a_1 \in A_1, \dots, a_n \in A_n, a_i - a_j \notin S_{ij} \text{ if } i \neq j\}, \quad (1.1)$$

where S_{ij} ($1 \leq i \neq j \leq n$) are finite subsets of F . Hou and Sun [11] established the following result.

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Theorem 1.1 (Hou-Sun). *Let C be given by (1.1) with $|S_{ij}| = m(1 \leq i \neq j \leq n)$. If $|A_1| = \cdots = |A_n| = k$ and $p(F) > \max\{n(k-1) - mn(n-1), mn\}$, then*

$$|C| \geq n(k-1) - mn(n-1) + 1.$$

The first result of this paper is to refine Theorem 1.1.

Theorem 1.2. *Let $S_{ij}(1 \leq i \neq j \leq n)$ be finite subsets of F with cardinality m , and let C be defined in (1.1). Suppose that $|A_j| \in \{k, k+1\}$ for $1 \leq j \leq n$. If $p(F) > \max\{mn, \sum_{j=1}^n (|A_j| - 1) - mn(n-1)\}$, then*

$$|C| \geq \sum_{j=1}^n (|A_j| - 1) - mn(n-1) + 1. \quad (1.2)$$

It was pointed out by Hou and Sun [11] that when $p(F) \leq n(k-1) - mn(n-1)$, Theorem 1.1 only implies

$$|C| \geq n \left\lfloor \frac{p(F) - 1}{n} \right\rfloor + 1, \quad (1.3)$$

where $\lfloor \alpha \rfloor$ denotes the greatest integer not exceeding real number α . Now we can deduce the following result from Theorem 1.2.

Theorem 1.3. *Suppose that $|S_{ij}| = m(1 \leq i \neq j \leq n)$, $|A_1| = \cdots = |A_n| = k$ and $p(F) > mn$. Let C be defined in (1.1). We have*

$$|C| \geq \min \{p(F), n(k-1) - mn(n-1) + 1\}.$$

When $p(F) \leq n(k-1) - mn(n-1)$, Theorem 1.3 implies $|C| \geq p(F)$, which improves upon the inequality (1.3). Dias da Silva and Hamidoune [5] showed that if $|A| > \sqrt{4p-7}$ then any element of $\mathbb{Z}/p\mathbb{Z}$ is a sum of $\lfloor \frac{|A|}{2} \rfloor$ distinct elements of A . We extend this result with the help of Theorem 1.3.

Theorem 1.4. *Let p be a prime and S be a subset of $F = \mathbb{Z}/p\mathbb{Z}$ with $|S| = m$. Let A be a subset of F with $|A| \geq \sqrt{4mp + 4m(m-3)} + 2 - m + 1$. Then any element of F can be written in the form $a_1 + \cdots + a_n$ with $n = \lfloor \frac{|A|-1+m}{2m} \rfloor$ and $a_i - a_j \notin S$ if $1 \leq i \neq j \leq n$.*

Section 2 is devoted to some preparations. The proofs of Theorems 1.2-1.3 will be given in Section 3. Finally, we deduce Theorem 1.4 from Theorem 1.3.

2. PRELIMINARIES

For a polynomial $g(x_1, \dots, x_n)$ over a field F , by $[x_1^{k_1} \cdots x_n^{k_n}]g(x_1, \dots, x_n)$ we mean the coefficient of the monomial $x_1^{k_1} \cdots x_n^{k_n}$ in $g(x_1, \dots, x_n)$. One has the following tool of the polynomial method. The reader may refer to the book of Tao and Vu [17, pp. 329-345] for the explanation of the polynomial method.

Lemma 2.1 ([1],[3]). *Let A_1, \dots, A_n be non-empty finite subsets of a field F , and let $P(x_1, \dots, x_n) \in F[x_1, \dots, x_n] \setminus \{0\}$. Suppose that $\deg P \leq \sum_{j=1}^n (|A_j| - 1)$. If*

$$[x_1^{|A_1|-1} \cdots x_n^{|A_n|-1}]P(x_1, \dots, x_n)(x_1 + \cdots + x_n)^{\sum_{j=1}^n (|A_j|-1) - \deg P} \neq 0,$$

then

$$|\{a_1 + \dots + a_n : a_1 \in A_1, \dots, a_n \in A_n, P(a_1, \dots, a_n) \neq 0\}| \geq \sum_{j=1}^n (|A_j| - 1) - \deg P + 1.$$

For nonnegative integers a_0, a_1, \dots, a_n , F. J. Dyson [6] in 1962 conjectured that the constant term of $\prod_{0 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^{a_j}$ is $\frac{(a_0 + \dots + a_n)!}{a_0! \dots a_n!}$. This conjecture was proved independently by Gunson [10] and by Wilson [18]. I. J. Good [9] in 1970 used the Lagrange interpolation formula to provide a short proof. D. Zeilberger [19] gave a combinatorial proof of Dyson's conjecture in the following equivalent form

$$[x_0^{a_0} \dots x_n^{a_n}] \prod_{0 \leq i < j \leq n} (x_i - x_j)^{a_i + a_j} = (-1)^{\sum_{j=0}^n (j+1)a_j} \frac{(a_0 + \dots + a_n)!}{a_0! \dots a_n!}.$$

Aomoto [4] proved that the constant term of

$$\prod_{l=1}^n (1 - \frac{x_l}{x_0})^{a + \chi(l \leq s)} (1 - \frac{x_0}{x_l})^b \prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^m$$

is

$$\prod_{l=0}^{n-1} \frac{(a + b + ml + \chi(l \geq n - s))! (ml + m)!}{(a + ml + \chi(l \geq n - s))! (ml + b)! m!},$$

where $\chi(l \geq t) = 1$ if $l \geq t$, and $\chi(l \geq t) = 0$ otherwise.

We can deduce the following result from Aomoto's identity.

Proposition 2.2. *Let $m \in \mathbb{N}$ and $k, n \in \mathbb{Z}^+$. Suppose that $k_j \in \{k, k+1\}$ for $1 \leq j \leq n$. If $k > m(n-1)$, then we have*

$$\begin{aligned} & [x_1^{k_1-1} \dots x_n^{k_n-1}] \prod_{1 \leq i \neq j \leq n} (x_i - x_j)^m (x_1 + \dots + x_n)^{\sum_{j=1}^n (k_j-1) - mn(n-1)} \\ &= \left(\prod_{j=0}^{s-1} \frac{1}{k-jm} \right) \frac{(\sum_{j=1}^n k_j - mn^2 + mn - n)!}{(m!)^n} \prod_{j=1}^n \frac{(jm)!}{(k-1-jm+m)!}, \end{aligned}$$

where $s = |\{1 \leq j \leq n : k_j = k+1\}|$.

In order to prove Proposition 2.2, we also need the following result (see Lemma 2.1 and Corollary A.1 in [8]).

Lemma 2.3 (Gessel-Lv-Xin-Zhou). *Let a_0, a_1, \dots, a_n be nonnegative integers, and let $L(x_1, \dots, x_n)$ be a Laurent polynomial independent of a_0 . Then the constant term of*

$$\prod_{l=1}^n (1 - \frac{x_l}{x_0})^{a_0} (1 - \frac{x_0}{x_l})^{a_l} L(x_1, \dots, x_n)$$

is a polynomial in a_0 for fixed a_1, \dots, a_n and the leading coefficient of such polynomial in a_0 coincides with the constant term of

$$\frac{1}{(a_1 + \dots + a_n)!} (x_1 + \dots + x_n)^{a_1 + \dots + a_n} \prod_{l=1}^n x_l^{-a_l} L(x_1, \dots, x_n).$$

Proof of Proposition 2.2. Without loss of generality, we assume that $k_1 = \dots = k_s = k + 1$ and $k_{s+1} = \dots = k_n = k$ for some s . For nonnegative integers a, b , we define $f(a; b, s, m)$ to be the constant term of $\mathcal{F}(x_0, x_1, \dots, x_n)$, where

$$\mathcal{F}(x_0, x_1, \dots, x_n) = \prod_{l=1}^n \left(1 - \frac{x_l}{x_0}\right)^a \left(1 - \frac{x_0}{x_l}\right)^{b+\chi(l \leq s)} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^m.$$

On substituting $x_l = y_l^{-1}$ for $0 \leq l \leq n$, we get

$$\mathcal{F}(y_0^{-1}, y_1^{-1}, \dots, y_n^{-1}) = \mathcal{G}(y_0, y_1, \dots, y_n),$$

where

$$\mathcal{G}(y_0, y_1, \dots, y_n) = \prod_{l=1}^n \left(1 - \frac{y_0}{y_l}\right)^a \left(1 - \frac{y_l}{y_0}\right)^{b+\chi(l \leq s)} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{y_j}{y_i}\right)^m.$$

We observe

$$\mathcal{G}(y_0, y_1, \dots, y_n) = \prod_{l=1}^n \left(1 - \frac{y_l}{y_0}\right)^{b+\chi(l \leq s)} \left(1 - \frac{y_0}{y_l}\right)^a \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{y_i}{y_j}\right)^m. \quad (2.1)$$

One can see that $f(a; b, s, m)$ is equal to the constant term of $\mathcal{G}(y_0, y_1, \dots, y_n)$. By (2.1) and Aomoto's identity, we conclude

$$f(a; b, s, m) = \prod_{l=0}^{n-1} \frac{(a + b + ml + \chi(l \geq n - s))!(ml + m)!}{(ml + a)!(b + ml + \chi(l \geq n - s))!m!}. \quad (2.2)$$

Then applying Lemma 2.3 with $a_0 = a$, $a_l = b + \chi(l \leq s)$ for $1 \leq l \leq n$ and $L(x_1, \dots, x_n) = \prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^m$, the leading coefficient of $f(a; b, s, m)$ in a_0 coincides with the constant term of

$$\frac{1}{(a_1 + \dots + a_n)!} (x_1 + \dots + x_n)^{a_1 + \dots + a_n} \prod_{l=1}^n x_l^{-a_l} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^m.$$

Note that

$$\begin{aligned} & (x_1 + \dots + x_n)^{a_1 + \dots + a_n} \prod_{l=1}^n x_l^{-a_l} \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^m \\ &= (x_1 + \dots + x_n)^{a_1 + \dots + a_n} \prod_{l=1}^n x_l^{-(a_l + (n-1)m)} \prod_{1 \leq i \neq j \leq n} (x_i - x_j)^m. \end{aligned}$$

The constant term of $(x_1 + \dots + x_n)^{a_1 + \dots + a_n} \prod_{l=1}^n x_l^{-a_l} \prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^m$ is

$$\left[\prod_{l=1}^n x_l^{a_l + (n-1)m} \right] \prod_{1 \leq i \neq j \leq n} (x_i - x_j)^m (x_1 + \dots + x_n)^{a_1 + \dots + a_n}.$$

By (2.2), the leading coefficient of $f(a; b, s, m)$ in a is

$$\prod_{l=0}^{n-1} \frac{(ml + m)!}{(b + ml + \chi(l \geq n - s))!m!}.$$

Now we can conclude that

$$\begin{aligned} & \left[\prod_{l=1}^n x_l^{a_l + (n-1)m} \right] \prod_{1 \leq i \neq j \leq n} (x_i - x_j)^m (x_1 + \cdots + x_n)^{a_1 + \cdots + a_n} \\ &= \frac{(a_1 + \cdots + a_n)!}{(m!)^n} \prod_{l=0}^{n-1} \frac{(ml + m)!}{(b + ml + \chi(l \geq n - s))!}. \end{aligned} \quad (2.3)$$

On substituting $b = k - (n - 1)m - 1$, we get the desired result from (2.3). The proof is completed. \square

3. PROOFS OF THEOREMS 1.2-1.4

Proof of Theorem 1.2. Since (1.2) holds trivially if $\sum_{j=1}^n (|A_j| - 1) - mn(n - 1) < 0$. Below we assume that $\sum_{j=1}^n (|A_j| - 1) \geq mn(n - 1)$. Define

$$P(x_1, \dots, x_n) = \prod_{1 \leq i \neq j \leq n} \prod_{s \in S_{ij}} (x_i - x_j - s).$$

We observe that $\deg(P) = mn(n - 1)$, and

$$C = \{a_1 + \cdots + a_n : a_1 \in A_1, \dots, a_n \in A_n, P(a_1, \dots, a_n) \neq 0\}.$$

Our objective is to prove

$$[x_1^{|A_1|-1} \cdots x_n^{|A_n|-1}] P(x_1, \dots, x_n) (x_1 + \cdots + x_n)^{\sum_{j=1}^n (|A_j|-1) - \deg P} \neq 0. \quad (3.1)$$

Then the desired conclusion follows from Lemma 2.1. Note that

$$\begin{aligned} & [x_1^{|A_1|-1} \cdots x_n^{|A_n|-1}] P(x_1, \dots, x_n) (x_1 + \cdots + x_n)^{\sum_{j=1}^n (|A_j|-1) - \deg P} \\ &= [x_1^{|A_1|-1} \cdots x_n^{|A_n|-1}] \prod_{1 \leq i \neq j \leq n} (x_i - x_j)^m (x_1 + \cdots + x_n)^{\sum_{j=1}^n (|A_j|-1) - mn(n-1)}. \end{aligned}$$

Let $s = \{1 \leq j \leq n : |A_j| = k + 1\}$. By Proposition 2.2,

$$\begin{aligned} & [x_1^{|A_1|-1} \cdots x_n^{|A_n|-1}] P(x_1, \dots, x_n) (x_1 + \cdots + x_n)^{\sum_{j=1}^n (|A_j|-1) - \deg P} \\ &= \left(\prod_{j=0}^{s-1} \frac{1}{k - jm} \right) \frac{(\sum_{j=1}^n |A_j| - mn^2 + mn - n)!}{(m!)^n} \prod_{j=1}^n \frac{(jm)!}{(k - 1 - jm + m)!}. \end{aligned}$$

On recalling the condition $p(F) > \max\{mn, \sum_{j=1}^n (|A_j| - 1) - mn(n - 1)\}$, we have established (3.1). The proof of Theorem 1.2 is completed. \square

Proof of Theorem 1.3. In view of Theorem 1.1, we only need to consider the case $p(F) \leq n(k - 1) - mn(n - 1)$. Set $k' = \lfloor \frac{p(F)-1}{n} \rfloor + m(n - 1) + 1$. Then there exist subsets $A'_j \subseteq A_j$ for $1 \leq j \leq n$ such that

$$\sum_{j=1}^n (|A'_j| - 1) - mn(n - 1) = p(F) - 1 \quad \text{and} \quad |A'_j| \in \{k', k' + 1\}.$$

We easily see that

$$C \supseteq \{a_1 + \cdots + a_n : a_1 \in A'_1, \dots, a_n \in A'_n, a_i - a_j \notin S_{ij} \text{ if } i \neq j\}.$$

Then we apply Theorem 1.2 to deduce that

$$\begin{aligned} |C| &\geq |\{a_1 + \cdots + a_n : a_1 \in A'_1, \dots, a_n \in A'_n, a_i - a_j \notin S_{ij} \text{ if } i \neq j\}| \\ &\geq \sum_{j=1}^n (|A'_j| - 1) - mn(n-1) + 1 \\ &= p(F) = \min \{p(F), n(k-1) - mn(n-1) + 1\}. \end{aligned}$$

This completes the proof. \square

Proof of Theorem 1.4. Note that since $n = \lfloor \frac{|A|-1+m}{2m} \rfloor$, we have $|A| - 1 + m \geq 2mn$. Since A is a subset of $F = \mathbb{Z}/p\mathbb{Z}$, we deduce that $p \geq |A| > 2mn - m \geq mn$. The conclusion of Theorem 1.4 is equivalent to

$$\{a_1 + \cdots + a_n : a_j \in A, a_i - a_j \notin S \text{ if } i \neq j\} = \mathbb{Z}/p\mathbb{Z}.$$

By Theorem 1.3, it suffices to prove

$$p \leq n(|A| - 1) - mn(n-1) + 1.$$

Let $r = 2m(\frac{|A|-1+m}{2m} - n)$. We can see that $|A| - 1 + m = 2mn + r$ and $0 \leq r \leq 2m - 1$. Then we obtain

$$(|A| - 1 + m)^2 = (2mn + r)^2 = r^2 + 4m(mn^2 + nr) \equiv r^2 \pmod{4m}. \quad (3.2)$$

In view of the condition $|A| \geq \sqrt{4mp + 4m(m-3) + 2} - m + 1$, we have

$$(|A| - 1 + m)^2 \geq 4mp + 4m(m-3) + 2 = 4m(p-1) + (2m-1)^2 - (4m-1).$$

Therefore,

$$(|A| - 1 + m)^2 \geq 4m(p-1) + r^2 - (4m-1). \quad (3.3)$$

It follows from (3.2) and (3.3) that $(|A| - 1 + m)^2 \geq 4m(p-1) + r^2$. This implies

$$n(|A| - 1) - mn(n-1) + 1 = \frac{(|A| - 1 + m)^2 - r^2}{4m} + 1 \geq p.$$

We complete the proof of Theorem 1.4. \square

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